Theoretisch Physik 1 Mechanik - Pankratov

Vorlesungsmitschrift

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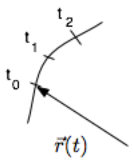
Inhaltsverzeichnis

Teil I
Vorlesung vom 16. Oktober 2007

1 Newtonsche Mechanik



Radius-Vektor $\vec{r}(t)$

$$m\ddot{\vec{r}}(t) = \vec{F}\left(\vec{r}(t), \dot{\vec{r}}(t), \ddot{\vec{r}}(t), t\right)$$

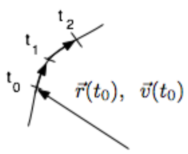
$$\vec{v}(t) = \dot{\vec{r}}(t)$$

$$m\frac{d}{dt}\vec{v}(t) = \vec{F}$$

$$\vec{v}(t) = \vec{v}(t_0) + \frac{1}{m}\int_{t_0}^{t_1} \vec{F}\left(\vec{r}(t), ...\right) dt$$

$$\vec{r}(t) = \vec{r}(t_o) + \int_{t_0}^{t_1} \vec{v}(t)dt$$

Schritt für Schritt:



$$\vec{v}(t_1) = \vec{v}(t_0) + \frac{\vec{F}\left(\vec{r}(t_0), \dot{\vec{r}}(t_1), ...\right)}{m}(t_1 - t_0)$$

$$\vec{r}(t_1) = \vec{r}(t_0 + \vec{v}(t_0)(t_1 - t_0)$$

$$\vec{v}(t_2) = \vec{v}(t_1) + \frac{\vec{F}\left(\vec{r}(t_1), ...\right)}{m}(t_2 - t_1)$$

$$\vec{r}(t_2) = \vec{r}(t_1 + \vec{v}(t_1)(t_2 - t_1)$$

$$\vdots$$

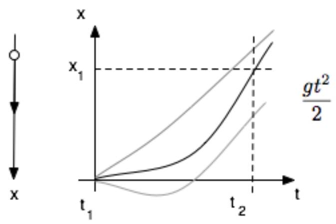
1.1 Koordinaten

$$m\ddot{\vec{r}} = \vec{F} \quad , \quad v_\tau = l \cdot \dot{\varphi} \quad , \quad \dot{v_\tau} = l \cdot \ddot{\varphi}$$

$$m\dot{v}_\tau = -mg\sin\varphi \quad , \quad ml\ddot{\varphi} + mg\sin\varphi = 0$$

$$\boxed{\ddot{\varphi} + \tfrac{g}{l}\sin\varphi = 0}$$

1.2 Freier Fall



1.2.1 Analogie: Fermatsches Prinzip

Lichtausbreitung: Brechungsindex $n(\vec{r})$

$$v(\vec{r}) = \frac{c}{n(\vec{r})} \quad , \quad n(\vec{r}) = const.$$

$$T[c_{12}] = T[y(x)] := \text{ Funktional}$$

$$T = \int_{c_{12}} \frac{dl}{v(\vec{r})} = \frac{1}{v}\int_{c_{12}} dl = \frac{1}{v} c_{12}$$

$$\min T[y(x)] = T^{\text{richtig}}[y^{richtig}]$$

$$dl = \sqrt{dx^2 + dy^2} = dy\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\sqrt{1 + (y'(x))^2}$$

$$T[y(x)] = \frac{1}{v}\int_{x_1}^{x_2} \sqrt{1 + (y'(x'))^2} dx \quad ; \quad \min T[y(x)] = T[y = const.]$$

1.2.2 Beispiel: Spiegelung

$$T[x] ist T(x)$$

c_{12} ist durch x parametrisiert

$$cT(x) = \sqrt{y_1^2 + x^2} + \sqrt{y_2^2 + (l - x)^2}$$

$$c\frac{dT}{dx} = \frac{x}{\sqrt{y_2^2 + x^2}} - \frac{(l - x)}{\sqrt{y_2^2 + (l - y)^2}} = \cos\theta_1 - \cos\theta_2 \quad \overset{\min}{\Rightarrow} \quad \theta_1 = \theta_2$$

1.3 Eindimensionales Potenzial

Funktional $S = [x(t)] = \int_{t_1}^{t_2} \left[\frac{m\dot{x}^2(t)}{2} - U(x(t))\right] dt$

$$\min S[x(t)] = S[x^{\text{richtig}}(t)]$$

Variation des Funktionals $S[x(t)] \to \delta S$

$$\delta S = S[x(t) + \delta x(t)] - S[x(t)] = 0 \quad \Rightarrow \quad \text{minimal-Prinzip}$$

$$\delta S = \int_{t_1}^{t_2} \frac{m}{2}[(\dot{x} + \delta\dot{x})^2 - \dot{x}^2]dt = \int_{t_1}^{t_2} [U(x + \delta x) - U(x)]dt =$$

$$= \underbrace{m \int_{t_1}^{t_2} \dot{x}\delta\dot{x}dt}_{\int_{t_1}^{t_2} \dot{x}d\delta x = \underbrace{\dot{x}\delta x|_{t_1}^{t_2}}_{=0} - \int_{t_1}^{t_2} \delta x(t)\ddot{x}dt} - \int_{t_1}^{t_2} \frac{dU}{dx}\delta x dt = 0 \quad \forall \ \delta x(t) \quad ; \quad m\ddot{x} = -\frac{dU}{dx}$$

1.4 Das Prizip der kleinsten Wirkung

1.4.1 verallgemeinerte Koordinaten

$$q_1, q_2, ..., q_s := \{q\} \quad q_i \ , \ i = 1 \to s$$

$$\dot{q}_1, \dot{q}_2, ..., \dot{q}_s$$

1.4.2 Lagrange-Funktion

$$\mathcal{L}(q, \dot{q}, t)$$

1.4.3 Wirkung-Funktion

$$S[q(t)] = \int_{t_1}^{t_2} \mathcal{L}(q, \dot{q}, t)dt$$

$$: \min S[q(t)] \quad \Rightarrow \quad S[q^{\text{richtig}}(t)]$$

$$\delta S = S[q + \delta q(t)] - S[q] = \int_{t_1}^{t_2} [\mathcal{L}(q_1 + \delta q, \dot{q} + \delta\dot{q}, t) - \mathcal{L}(q, \dot{q}, t)]dt =$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q}\delta q + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta\dot{q}}_{\int \frac{\partial \mathcal{L}}{\partial \dot{q}} d\delta q = \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q|_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta q(t) \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} dt \qquad \forall \delta q(t)} \right) dt =$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial q} \right) \delta q(t)dt = 0$$

1.4.4 Lagrange-Gleichungen

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

1D-Fall $\quad \mathcal{L}(x, \dot{x}, t) = \frac{m\dot{x}^2}{2} - U(x) \qquad q \equiv x$

$$\frac{d}{dt}m\dot{x} + \frac{dU}{dx} = 0 \quad , \quad -\frac{dU}{dx}$$

$$(\phi, \dot{\phi}) = (q, \dot{q})$$

$$\mathcal{L}(\phi, \dot{\phi}) = \frac{m}{\vec{r}}(l\dot{\phi}^2) + (rmgl\cos\phi)$$

$$\frac{d}{dt}ml\phi^2\dot{\phi} - mgl\sin\phi = 0$$

$$\ddot{\phi} + \frac{g}{l}\sin\phi = 0$$

Teil II
Vorlesung vom 18. Oktober 2007

2 Lagrange Mechanik

- Verallgemeinerte Koordinaten $q_1, q_2, ..., q_s = \{g\}$

  $s =$ Zahl der Freiheitsgrade

- Lagrange-Funktion $\mathcal{L}(q(t), \dot{q}(t), t)$
- Das Prinzip der kleinsten Wirkung

  $$S[q(t)] = \int_{t_1}^{t_2} \mathcal{L}(q(t), \dot{q}(t), t) dt$$

  $$\min S[q(t)] = S[q^{\text{richtig}}(t)]$$
  $$S[q + \delta q] - S[q] = \mathcal{O}(\delta q^2)$$

- Lagrange-Gleichungen

  $$\boxed{\tfrac{d}{dt}\tfrac{\partial \mathcal{L}}{\partial \dot{q}_i} - \tfrac{\partial \mathcal{L}}{\partial q_i} = 0} \qquad ; \quad \mathcal{L}(q, \dot{q}, t)$$

  Anfangsbedingungen: $q_i(t_1), \dot{q}_i(t_1)$

2.1 Die Lagrange-Funktion

$$\mathcal{L}(x, \dot{x}, t) = \frac{m\dot{x}^2}{2} = U(x) \quad ; \quad \frac{d}{dt} m\dot{x} + \frac{dU}{dx} = 0 \quad ; \quad m\ddot{x} = -\frac{dU}{dx} = \dot{F}(x)$$

2.2 Langrange-Funktion von Grundprinzipien

2.2.1 Freies Teilchen

$$\mathcal{L}(\vec{r}, \dot{\vec{r}}, t)$$

Es gibt inertiale Bezugssysteme

- Raum ist homogen $\Rightarrow \quad \mathcal{L}(\dot{\vec{r}}, t)$
- Isotrop $\Rightarrow \quad \mathcal{L}(\vec{v}^2, t)$
- Zeit ist homogen $\mathcal{L}(\vec{v}^2, t)$

2.2.2 Lagrange-Gleichung

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \vec{v}} = 0$$
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \vec{v}^2}\frac{\partial \vec{v}^2}{\partial \vec{v}} = 2\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \vec{v}^2}\right) = 0 \quad \Rightarrow \quad \vec{v}(t) = const.$$

Galileisches Inertialgesetzt

2.2.3 Galileisches Relativitätsprinzip

U und U' sind nicht unterscheidbar!

$$\vec{v}' = \vec{v} - \vec{u} \quad , \quad \vec{v} = \vec{v}' + \vec{u}$$

$$\vec{r}(t) = \vec{r}'(t) + \vec{v}(t) \quad \text{Galileische Transformation}$$

$\mathcal{L}$-Funktion in $U : L(\vec{v}^2)$
$\mathcal{L}$-Funktion in $U' : L\left((\vec{v} - \vec{u})^2\right)$

Lagrange-Funktion ist nicht ganz eindeutig!

$$\mathcal{L}'(q, \dot{q}, t) = \mathcal{L}(q, \dot{q}, t) + \frac{d}{dt} f(q, t)$$

$$S' = \int_{t_1}^{t_2} \mathcal{L}' dt = \int_{t_1}^{t_2} \mathcal{L} dt + \int_{t_1}^{t_2} \frac{d}{dt} f dt = S + f(t_2) - f(t_1)$$

$$\delta S' = \delta S$$

$$\mathcal{L}\left((\vec{v} - \vec{u})^2\right) \overset{?}{\leftrightarrow} \mathcal{L}(\vec{v}^2)$$

$$\lim \vec{v} \to 0; \quad (\vec{v} - \vec{u})^2 = \vec{v}^2 - 2\vec{v}\vec{u} + \mathcal{O}(\vec{u}^2)$$

...wird noch ergänzt...

Teil III

Vorlesung vom 23. Oktober 2007

2.3 Massenpunkt auf der Kegelfläche $(\phi, r; \dot{\phi}, \dot{r})$

$$\mathcal{L} = \frac{mv^2}{2} - U = \frac{m}{2}\left(r^2\dot{\phi}^2 + \frac{\dot{r}^2}{\sin^2\theta}\right) - \underbrace{mg\cot\theta\dot{r}}_{U(r)}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \frac{\partial\mathcal{L}}{\partial\phi} = 0; \quad \frac{d}{dt}\left(r^2\dot{\phi}\right) = 0; \quad r^2\dot{\phi} = M; \quad \dot{\phi} = \frac{M}{r^2}$$

$$\frac{\ddot{r}}{\sin^2\theta} - \underbrace{\frac{M^2}{r^3} + g\cot\theta}_{-F(r)=\frac{dU_{eff}(r)}{dr}} = 0 \quad ; \quad \frac{\ddot{r}}{\sin^2\theta} = F(r)$$

Effektivpotenzial $U_{eff}(r) = \int \left(g\cot\theta - \frac{M^2}{r^3}\right) dr$

$$\boxed{U_{eff}(r) = g\cot\theta \cdot r + \frac{M^2}{2r^3}}$$

2.3.1 1D-Bewegung in $U_{eff}(r)$

$$\frac{\ddot{r}}{\sin^2\theta} + \frac{dU_{eff}(r)}{dr} = 0$$

Periodische Bewegung

$$r(t) : r_{\min} \leq r(t) \leq r_{\max}$$

$$g\cot\theta r_m + \frac{M^2}{2r_m^2} = 0 \quad ; \quad r_m = r_{\max} \text{ oder } r_{\min} > 0$$

$r_0 :=$ Ruhepunkt

$$\left.\frac{dU_{eff}(r)}{dr}\right|_{r_0} = 0 \quad , \quad \frac{M^2}{r_0^3} + g\cot\theta = 0$$

2.3.2 $\phi(t)$-Koordinate

$$\dot{\phi} = \frac{M}{r^2(t)} \quad ; \quad r(t) = r_0 + \rho(t)$$

$$\dot{\phi} = \frac{M}{(r_0 + \rho(t))^2} \approx \frac{M}{r_0^2}$$

Laufbahn: $(r(t), \phi(t)) \Rightarrow \phi(t)$

2.3.3 Ist die Laufbahn geschlossen?

$$n\Delta\phi = 2\pi \cdot m$$

$$\boxed{\tfrac{\Delta\phi}{2} = \pi\tfrac{m}{n}}$$

$r(t)$ ausrechnen: 1D-Bewegung integrieren

$$\frac{\ddot{r}}{\sin^2\theta} + \frac{dU_{eff}(r)}{dr} = 0\ , \qquad \frac{\dot{r}\ddot{r}}{\sin^2\theta} + \dot{r}\frac{dU_{eff}}{dr} = 0$$

$$\frac{d}{dt}\frac{\dot{r}^2}{2\sin^2\theta} + \frac{d}{dt}U_{eff}(r) = 0$$

$$\frac{d}{dt}\left(\underbrace{\frac{\dot{r}^2}{2\sin^2\theta} + U_{eff}(r(t))}_{E}\right) = 0 \quad ; \quad \frac{\dot{r}^2}{2\sin^2\theta} + U_{eff}(r) = E = const.$$

$$\dot{r}(t) = \sqrt{2}\sin\theta\sqrt{E - U_{eff}(r)} = \frac{dr}{dt}$$

$$\int dt = \frac{1}{\sqrt{2}\sin\theta}\int\frac{dr}{\sqrt{E - U_{eff}(r)}}$$

$$\boxed{\frac{1}{\sqrt{2}\sin\theta}\int\frac{dr}{\sqrt{E - g\cot\theta r - \frac{M^2}{r^2}}}} \quad \Rightarrow \quad t(r),\ \ r(t)$$

$$\phi(t) = ?$$

$$\dot{\phi} = \frac{M}{r^2(t)} \quad : \quad \frac{d\phi}{dt}\frac{dr}{dr} = \frac{M}{r^2} \quad \Rightarrow \quad \frac{d\phi}{dr}\dot{r} = \frac{M}{r^2} \quad \text{bzw.} \quad \frac{d\phi}{dr} = \frac{M}{r^2\dot{r}}$$

$$\phi(r) = \int\left(\frac{\frac{M}{r^2}dr}{\sqrt{2}\sin\theta\sqrt{E - U_{eff}(r)}}\right)$$

$$\Delta\phi = \int_{r_{\min}}^{r_{\max}}\frac{\frac{M}{r^2}dr}{\sqrt{2}\sin\theta\sqrt{E - \underbrace{g\cot\theta r - \frac{M^2}{2r^2}}_{U_{eff}(r)}}}$$

$$\frac{\Delta\phi}{2\pi} = \frac{m}{n}$$

Potenzielle Energie: $U = g\cot\theta r$

$$U(x) = \alpha r^2 \text{ oder } \frac{\beta}{r}$$

3 Erhaltungssätze

3.1 Energieerhaltung

Zeit ist homogen: $\mathcal{L}(q, \dot{q})$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} - \frac{\partial\mathcal{L}}{\partial q_i} = 0$$

$$\begin{aligned}
\frac{d}{dt}L(q_i(t), \dot{q}_i(t)) &= \sum_i \left(\frac{\partial\mathcal{L}}{\partial q_i}\frac{\partial q_i}{t} + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\frac{\partial\dot{q}_i}{\partial t}\right) = \\
&= \sum_i \left[\underbrace{\left(\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right)\dot{q}_i + \left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\frac{\partial\dot{q}}{\partial t}\right)}_{\left(\frac{da}{dt}\right)b + a\left(\frac{db}{dt}\right) = \frac{d}{dt}(ab)}\right] = \\
&= \sum_i \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\dot{q}_i\right)
\end{aligned}$$

$$\frac{d}{dt}\underbrace{\left[\sum_i \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\dot{q}_i - \mathcal{L}\right]}_{\sum_i \frac{\partial\mathcal{L}(q,\dot{q})}{\partial\dot{q}_i} - \mathcal{L}(q(t),\dot{q}(t)) = E = const.} = 0 \quad \text{Unter Bedingung, dass } \mathcal{L}(q, \dot{q})$$

3.1.1 Beispiel: Kreiskegel

$$\begin{aligned}
E &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}} + \frac{\partial\mathcal{L}}{\partial\dot{r}}\dot{r} - \mathcal{L} = \\
&= m\left(r^2\dot{\phi}^2 + \frac{\dot{r}^2}{\sin^2\theta}\right) - \frac{1}{2}m(\quad) + mg\cot\theta r = \\
&= \frac{m}{2}\left(r^2\dot{\phi}^2 + \frac{r^2}{\sin\theta}\right) + mg\cot\theta r = \boxed{E = \frac{mv^2}{2} + U}
\end{aligned}$$

$E = \frac{mv^2}{2} + U$ ist kein Zufall!

$$\mathcal{L}(q_i, \dot{q}_i) = \sum_{i,j} a_{ij}\dot{q}_i\dot{q}_j - Uq_i$$

$$\begin{aligned}
\sum_k \frac{\partial\mathcal{L}}{\partial\dot{q}_k}\dot{q}_k - \mathcal{L} &= \sum_{i,j,k} a_{ij}(\delta_{ik}\dot{q}_j\dot{q}_k + \delta_{jk}\dot{q}_i\dot{q}_k) - \mathcal{L} = \\
&= \sum_{i,j} a_{ij}(\dot{q}_i\dot{q}_j + \dot{q}_j\dot{q}_i) - \mathcal{L} = \\
&= \sum_{i,j} a_{ij}\dot{q}_i\dot{q}_j + U(q_i) = \underbrace{T + U}_{E_{kin} + E_{pot}}
\end{aligned}$$

3.2 Impulserhaltung

$\mathcal{L}(\vec{r}_a, \dot{\vec{r}}_a)$

$$\begin{aligned} \delta\mathcal{L} &= \mathcal{L}(\vec{r}_a + \vec{\epsilon}, \dot{\vec{r}}_a) - \mathcal{L}(\vec{r}_a, \dot{\vec{r}}_a) = 0 \\ \delta\mathcal{L} &= \sum_a \frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}_a}\vec{\epsilon} = \vec{\epsilon}\sum_a \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}_a} = \\ &= \vec{\epsilon}\frac{d}{dt}\left(\sum_a \frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}_a}\right) = 0 \end{aligned}$$

Erhaltungsgröße: $\boxed{\vec{p} = \sum_a \frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}_a}}$:= Impuls $\quad \frac{d}{dt}\vec{p} = 0$

3.2.1 Mehrere Teilchen

$$\mathcal{L}(\vec{r}_1, ..., \vec{r}_N; \dot{\vec{r}}_1, ..., \dot{\vec{r}}_N) = \sum_a \frac{m\vec{v}^2}{2} - U(\vec{r}_1, ..., \vec{r}_N \qquad \dot{\vec{r}}_a = \vec{v}_a$$

$$\boxed{\vec{p} = \sum_a \frac{\partial\mathcal{L}}{\partial\vec{v}_a} = \sum_a m\vec{v}_a}$$

$$\frac{d\vec{p}}{dt} = \sum_a m\ddot{\vec{r}}_a = \sum_a \vec{F}_a = 0$$

Für zwei Teilchen: $\vec{F}_{12} + \vec{F}_{21} = 0 \quad ; \quad \vec{F}_{12} = -\vec{F}_{21}$

3.2.2 Verallgemeinerte Koordinaten

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\dot{q}_i} - \frac{\partial\mathcal{L}}{\partial q_i} = 0$$

- Verallgemeinerter Impuls: $p_i = \frac{\partial\mathcal{L}}{\partial\dot{q}_i}$
- Verallgemeinerte Kraft: $F_i = \frac{\partial\mathcal{L}}{\partial q_i}$
- Verallgemeinerter Newtonscher Satz: $\frac{d}{dt}p_i = F_i$

z.B.: Kegelfläche-Bewegung:

$$\mathcal{L}(\dot{\phi}, r, \dot{r}) = \frac{m}{2}\left(r^2\dot{\phi}^2 + \frac{\dot{r}^2}{\sin^2\theta}\right) - mg\cos\theta r$$

3.2.3 Verallgemeinerter Impuls

$$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mr^2\dot{\phi} \quad ; \quad \frac{d}{dt}p_\phi = 0 \quad ; \quad F_\phi = \frac{\partial\mathcal{L}}{\partial\phi} = 0$$

$$p_r = \frac{\partial\mathcal{L}}{\partial\dot{r}}$$

3.2.4 Impuls ist Bezugssystemabhängig

$$\vec{p} = \sum_a m\vec{v}_a$$

Galileische Transformation (Systeme $\mathcal{K}$ und $\mathcal{K}'$): $\vec{r}_a = \vec{r}'_a + \vec{v}t$, $\vec{v}_a = \vec{v}'_a + \vec{v}$

$$\vec{p} = \sum_a m_a\vec{v}_a = \sum_a m_a(\vec{v}'_a + \vec{v}) = \sum_a m_a\vec{v}'_a + \vec{v}\underbrace{\sum_a m_a}_{\mu}$$

$$\vec{p} = \vec{p}' + \mu\vec{v} \quad \Leftrightarrow \quad \vec{p} = \vec{p}' - \mu\vec{v}$$

Es ist immer möglich $\mathcal{K}'$ so zu wählen, dass $\vec{p}' = 0$

$$\vec{v}_S = \frac{\vec{p}}{\mu} = \frac{\sum_a m\vec{v}_a}{\sum_a m_a} = \frac{\sum_a m_a\dot{\vec{r}}_a}{\sum_a m_a} = \frac{d}{dt}\frac{\sum_a m_a\vec{r}_a}{\sum_a m_a} = \frac{d}{dt}\vec{R}_S(t)$$

wobei $\vec{R}_s(t) = \dfrac{\sum_a m_a\vec{r}_a}{\sum_a m_a}$

$\mathcal{K}'$ ist Schwerpunktsystem, $R_S(t)$ ist ein Schwerpunkt des Systems.

3.3 Galileische Transformation der Energie

Galileische Transformation (Systeme $\mathcal{K}$ und $\mathcal{K}'$): $\vec{r}_a = \vec{r}'_a + \vec{v}t$, $\vec{v}_a = \vec{v}'_a + \vec{v}$

$$E = \sum_a \frac{m_a\vec{v}_a^2}{2} + U(\vec{r}_1, ..., \vec{r}_N)$$

$$U(\vec{r}'_1, ..., \vec{r}'_N) = U(\vec{r}_1, ..., \vec{r}_N)$$

$$\begin{aligned} E &= \sum_a \frac{m_a}{2}(\vec{v}'_a + 2\vec{v}'^2_a\vec{v} + \vec{v}^2) + U = \\ &= \left(\underbrace{\sum_a \frac{m_a\vec{v}'^2_a}{2} + U}_{E=E'+\vec{p}'\vec{v}+\frac{\mu\vec{v}^2}{2}}\right) + \vec{v}\sum_a m_a\vec{v}'_a + \frac{\mu\vec{v}^2}{2} \end{aligned}$$

$$\boxed{E = E' + \vec{p}'\vec{v} + \frac{\mu\vec{v}^2}{2}}$$

Wenn $\mathcal{K}'$ ein Schwerpunktsystem ist ($\vec{p}' = \sum_a m_a\vec{v}'_a = 0$)

$$\boxed{E = E'_S + \frac{\mu\vec{v}^2}{2}}$$

E'_S := innere Energie des Systems

$$\underbrace{\text{Energie} E = const. \quad , \quad \text{Impuls} \vec{p} = const.}$$

zwei unabhängige Integrale der Bewegung (Erhaltungsgrößen)

3.3.1 Einfaches Beispiel: Freies Teilchen

$$E = \frac{m\vec{v}^2}{2} \quad , \quad \vec{p} = m\vec{v}$$

$$E(\vec{p}) = \frac{(m\vec{v})^2}{2m} = \frac{\vec{p}^3}{2m}$$

$$\epsilon(\vec{p}) = \hbar\omega = \hbar c|\vec{k}| = c|\vec{p}|$$

3.4 Drehimpulserhaltung

Raumisotropie

$$\vec{r}' = \vec{r} + \delta\vec{r} \quad \text{Drehvektor}\delta\vec{\phi}$$

$$\delta\vec{r} = \vec{e}_\phi r \sin\theta\delta\phi = (\delta\vec{\phi} \times \delta\vec{r})$$

Für jedes Teilchen $\vec{r}_a : \vec{r}_a + \delta\vec{r}_a = (\delta\vec{\phi} \times \vec{r}_a)$

$\mathcal{L}$-Funktion: $\mathcal{L}(\vec{r}_1, \vec{r}_2), \vec{r}_3, ..., \dot{\vec{r}}_1, \dot{\vec{r}}_2...$

$$\vec{r}_a \to \vec{r}_a + \delta\vec{r}_a$$

$$\delta\mathcal{L} = \mathcal{L}(\vec{r}_1 + \delta\vec{r}_1), \vec{r}_2 + \delta\vec{r}_2, ...) - \mathcal{L}(\vec{r}_1, \vec{r}_2) = 0$$

$$\begin{aligned} \delta\mathcal{L} &= \textstyle\sum_a \frac{\delta\mathcal{L}}{\delta\vec{r}_a}\delta\vec{r}_a = \\ &= \textstyle\sum_a \frac{d}{dt}\vec{P}_a(\delta\vec{\phi} \times \vec{r}_a) = \\ &= \textstyle\frac{d}{dt}\sum_a \underbrace{(\delta\vec{\phi} \times \vec{r}_a)\vec{p}_a}_{\text{Mischprodukt der Vektoren}} \\ &= \textstyle\frac{d}{dt}\sum_a \delta\vec{\phi}(\vec{r}_a \times \vec{p}_a) = \dagger \\ &= \underbrace{\delta\vec{\phi}\frac{d}{dt}\sum_a}(\vec{r}_a \times \vec{p}_a) = 0 \end{aligned}$$

Drehimpuls: $\vec{M} = \sum_a \vec{r}_a \times \vec{p}_a$, Impuls: $\vec{p} = \sum_a \vec{p}_a = \sum_a \vec{p}_a\vec{v}_a$
Wenn: $\delta\vec{\phi} = \vec{e}_z\delta\phi_z \Rightarrow \mathcal{L}|_z = \delta\phi_z\frac{d}{dt}M_z = 0$

$$\Rightarrow \frac{dM_z}{dt} = 0 \quad ; \quad \boxed{M_z = const.} \ \text{Erhaltungsgröße}$$

$$\delta\phi_{x,y} \to \mathcal{L}|_{x,y}, \quad M_y, M_z = const.$$

$$\Rightarrow \vec{M} = const.$$